Bosonization and Duality
of
Massive Thirring Model

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Abstract

   Starting from a reformulation of the Thirring model as a gauge theory, we consider the bosonization of the D-dimensional multiflavor massive Thirring model ($D \geq 2$) with four-fermion interaction of the current-current type. Our method leads to a novel interpolating Lagrangian written in terms of two gauge fields. Especially we pay attention to the case of very massive fermion $m \gg 1$ in (2+1) and (1+1) dimensions. Up to the next-to-leading order of $1/m$, we show that the (2+1)-dimensional massive Thirring model is mapped to the Maxwell-Chern-Simons theory and that the (1+1)-dimensional massive Thirring model is equivalent to the massive free scalar field theory. In the process of the bosonization of the Thirring model, we point out the importance of the gauge-invariant formulation. Finally we discuss a possibility of extending this method to the non-Abelian case.

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1 Introduction and main results

Recently the Thirring model [1] was reformulated as a gauge theory [2] and was identified with a gauge-fixed version of the corresponding gauge theory by introducing the Stückelberg field θ in addition to the auxiliary vector field A_μ. In this formulation, the auxiliary field A_μ is identified with the gauge field. In the previous paper [3], we have given another reformulation of the Thirring model as a gauge theory based on the general formalism for the constrained system, so called the Batalin-Fradkin-Vilkovisky (BFV) formalism [4]. Especially Batalin-Fradkin (BF) method [5] gives the general procedure by which the system with the second class constraint is converted to that with the first class one. The new field which is necessary to complete this procedure is called the Batalin-Fradkin field [5]. In the massive gauge theory the Batalin-Fradkin field is nothing but the well-known Stückelberg field as shown in [6].

   We consider the mapping from quantum field theory of interacting fermions onto an equivalent theory of interacting bosons. In this paper such an equivalent bosonic theory to the original *massive* Thirring model is obtained starting from the formulation of the D-dimensional Thirring model ($D = d + 1 \geq 2$) as a gauge theory. This is a kind of bosonization. The bosonization of the (1+1) dimensional Thirring model has been studied by many authors and is well known, see for example [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In this paper we consider the bosonization of the multiflavor massive Thirring model in $D = d + 1 \geq 2$ dimensions. Especially, we study the large fermion mass limit $m \gg 1$, in (2+1) and (1+1) dimensional cases explicitly. A motivation of studying the multiflavor case stems from the renormalizability of the Thirring model in $1/N$ expansion at least for $D = 3$ [17], although it is perturbatively non-renormalizable for $D > 2$.

   In the case of (2+1)-dimensions, we show that, up to the next-to-leading order in the inverse fermion mass, $1/m$, *the (2+1)-dimensional massive Thirring model is equivalent to the Maxwell-Chern-Simons theory, the topologically massive $U(1)$ gauge theory.* This equivalence in three dimensions has been shown, to the lowest order in $1/m$ for a massive Thirring model, by Fradkin and Schaposnik [18]. The fermi-bose equivalence was discussed earlier also in [19, 20]. In the multiflavor case we consider in this paper, there are many possibilities of taking the massive limits. Indeed we point out that there is a possibility that the leading order term, i.e., Chern-Simons term vanishes and hence only the non-local Maxwell-like term remains in the next-to-leading order, depending on the configuration of fermion masses. In such a case we are forced to consider the next-to-leading order of $1/m$. As discussed in the previous paper [3], the existence of the kinetic term for the gauge field A_μ is essential in formulating the Thirring model as a gauge theory based on the BFV formalism. Such a configuration of fermion masses has important implications from the viewpoint of chiral symmetry breaking for 4-component fermions[3]. This is in sharp contrast to the case treated in [18] where the non-vanishing Chern-Simons term is assumed from the beginning and hence only the leading order term is taken into account for bosonization. In our bosonization scenario the gauge invariance should be preserved in the step of calculations. In

this connection we mention the choice of appropriate regularization.

The method of Fradkin and Schaposnik is more elegant than ours. However we can raise the following questions on their treatment of bosonization.

1. The interpolating Lagrangian $\mathcal{L}_{FS}$ of Deser and Jackiw [21] was introduced as a device for showing this equivalence in [18]:

$$\mathcal{L}_{FS}[V_\mu, H_\mu] = \frac{1}{2}V^\mu V_\mu - \frac{1}{2}\epsilon^{\mu\nu\rho}V_\rho(\partial_\mu H_\nu - \partial_\nu H_\mu) + \frac{2\pi}{G}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu H_\rho. \quad (1.1)$$

However the origin of the interpolating Lagrangian was never shown. How can we derive this type of interpolating Lagrangian from the Thirring model or the Maxwell-Chern-Simons theory?

2. The original Thirring model has no gauge symmetry. Nevertheless the equivalent bosonic theory, the Maxwell-Chern-Simons theory, has $U(1)$ gauge symmetry. Where does this gauge symmetry come from?

3. The interpolating Lagrangian is invariant under a gauge transformations for the gauge field H_μ, while the auxiliary vector field V_μ does not have the gauge symmetry: $\mathcal{L}_{FS}$ is invariant under $\delta H_\mu = \partial_\mu\lambda$ and $\delta V_\mu = 0$. In integrating out the the gauge field H_μ in the interpolating Lagrangian, they had to introduce the gauge-fixing term ad hoc.

In contrast to [18], we start from the gauge-invariant or more precisely the BRS-invariant formulation of the Thirring model and discuss the bosonization of the massive Thirring model. The bosonization of the massless Thirring model was already discussed based on this strategy in [2]. However the bosonization in the usual or exact sense is possible only in the massive case except for $D = 2$, as shown in this paper. Our approach is more direct than the method of [18] and is able to *derive* a novel interpolating Lagrangian as a natural consequence (in an intermediate step) of bosonization. Our interpolating Lagrangian is written in terms of two gauge fields which have independent gauge symmetries. Thanks to the gauge invariant formulation, we can fix definitely the gauge invariance appearing in the interpolating Lagrangian [21]. It is interesting to extend our method into the non-Abelian case [21, 22]. This issue will be discussed in the final section.

The (1+1)-dimensional case is also discussed, although the gauge symmetry disappears in this case. The Thirring model in (1+1)-dimensions is rewritten into the equivalent scalar field theory [7, 8, 9]. It is well known that the (1+1)-dimensional *massless* Thirring model is exactly solvable in the sense that the model is equivalent to the massless *free* scalar theory [1]. In this paper we show, as a special case of the above formalism, *the massive Thirring model in (1+1)-dimensions is equivalent to the massive free scalar field theory in (1+1)-dimensions*, up to the next-to-lowest order in $1/m$. This is consistent with the well-known result.

In the previous paper [3] we have studied the spontaneous breakdown of the chiral symmetry in the massless Thirring model, $m \to 0$. In this paper we consider another extreme limit $m \to \infty$. According to the universality hypothesis,

the critical behavior of the model will be characterized by a small number of parameters appearing in the original Lagrangian of the model such as symmetry, range of interaction and dimensionality. Therefore, in the large fermion mass limit, the critical behavior of the Thirring model will be characterized by studying the equivalent bosonic theory according to the above bosonization.

2 Bosonization

The Lagrangian of the D-dimensional multiflavor Thirring model ($D = d+1 \geq 2$) is given by

$$\mathcal{L}_{Th} = \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j - m_j \bar{\psi}^j \psi^j - \frac{G}{2N}(\bar{\psi}^j \gamma_\mu \psi^j)(\bar{\psi}^k \gamma^\mu \psi^k), \tag{2.1}$$

where ψ^j is a Dirac spinor and the indices j, k are summed over from 1 to N, and the gamma matrices $\gamma_\mu (\mu = 0, ..., D-1)$ are defined so as to satisfy the Clifford algebra, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1} = 2\text{diag}(1, -1, ..., -1)$.

By introducing an auxiliary vector field A_μ, this Lagrangian is equivalently rewritten as

$$\mathcal{L}_{Th'} = \bar{\psi}^j i\gamma^\mu(\partial_\mu - i\frac{g}{\sqrt{N}}A_\mu)\psi^j - m_j \bar{\psi}^j \psi^j + \frac{M^2}{2}A_\mu^2, \tag{2.2}$$

where we have introduced a parameter $M(= 1)$ with the dimension of mass, $dim[m] = dim[M] = 1$ and put $G = \frac{g^2}{M^2}$. Thanks to the parameter M, all the fields have the corresponding canonical dimensions: $dim[\bar{\psi}] = dim[\psi] = (D-1)/2, dim[A_\mu] = (D-2)/2$ and then the coupling constant has the dimension: $dim[g] = (4-D)/2, dim[G] = 2 - D$.

The theory with this Lagrangian is identical with that of the massive vector field with which the fermion couples minimally, since a kinetic term for A_μ is generated through the radiative correction although it is absent originally. As is known from the study of massive vector boson theory [6], the Thirring model with the Lagrangian (2.2) is cast into the form which is invariant under the Becchi-Rouet-Stora (BRS) transformation by introducing an additional field θ. The field θ is called the Stückelberg field and identified with the Batalin-Fradkin (BF) field [5] in the general formalism for the constrained system [4]. Then we start from the Lagrangian with covariant gauge-fixing [2, 3]:

$$\mathcal{L}_{Th''} = \bar{\psi}^j i\gamma^\mu(\partial_\mu - i\frac{g}{\sqrt{N}}A_\mu)\psi^j - m_j \bar{\psi}^j \psi^j + \frac{M^2}{2}(A_\mu - \sqrt{N}M^{-1}\partial_\mu \theta)^2$$

$$-A_\mu \partial^\mu B + \frac{\xi}{2}B^2 + i\partial^\mu \bar{C}\partial^\mu C. \tag{2.3}$$

Actually this Lagrangian is invariant under the BRS transformation:

$$\delta_B A_\mu(x) = \partial_\mu C(x),$$

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$$
\begin{aligned}
\delta_B B(x) &= 0, \\
\delta_B C(x) &= 0, \\
\delta_B \bar{C}(x) &= iB(x), \\
\delta_B \theta(x) &= \frac{M}{\sqrt{N}} C(x), \\
\delta_B \psi^j(x) &= \frac{ig}{\sqrt{N}} C(x)\psi^j(x),
\end{aligned}
\tag{2.4}
$$

where $C(x)$ and $\bar{C}(x)$ are ghost fields, and $B(x)$ is the Nakanishi-Lautrap Lagrange multiplier field.

First we consider the case of $D \geq 3$. The two-dimensional case is discussed separately in section 4. By introducing an auxiliary vector field f_μ, the "mass term" of the gauge field is linearized: For $K_\mu = \sqrt{N}M^{-1}\partial_\mu\theta$,

$$
\begin{aligned}
& \int \mathcal{D}\theta \exp\left\{ i \int d^D x \frac{1}{2} M^2 (A_\mu - K_\mu)^2 \right\} \\
=\ & \int \mathcal{D}\theta \int \mathcal{D}f_\mu \exp\left\{ i \int d^D x \left[-\frac{1}{2} f_\mu f^\mu + M f^\mu (A_\mu - K_\mu) \right] \right\} \\
=\ & \int \mathcal{D}f_\mu \delta(\partial^\mu f_\mu) \exp\left\{ i \int d^D x \left[-\frac{1}{2} f_\mu f^\mu + M f^\mu A_\mu \right] \right\},
\end{aligned}
\tag{2.5}
$$

where in the last step we have integrated out the scalar mode θ.

Applying the Hodge decomposition [23] [1] to the 1-form f_μ, we see f_μ is written as

$$
f_{\mu_1} = \partial_{\mu_1}\phi + \epsilon_{\mu_1 \ldots \mu_D} \partial^{\mu_2} H^{\mu_3 \ldots \mu_D},
\tag{2.6}
$$

where we have introduced the antisymmetric tensor field $H_{\mu_1 \ldots \mu_{D-2}}$ of rank $D-2$, which satisfies the Bianchi identity. Since f_μ is divergence free, $\partial^\mu f_\mu = 0$, we can put $f_{\mu_1} = \epsilon_{\mu_1 \ldots \mu_D} \partial^{\mu_2} H^{\mu_3 \ldots \mu_D}$. Then we have

$$
\begin{aligned}
& \int \mathcal{D}\theta \exp\left\{ i \int d^D x \frac{1}{2} M^2 (A_\mu - K_\mu)^2 \right\} \\
=\ & \int \mathcal{D}H_{\mu_1 \ldots \mu_{D-2}} \exp\Big\{ i \int d^D x \Big[\\
& \frac{(-1)^D}{2(D-1)} \tilde{H}_{\mu_1 \ldots \mu_{D-1}} \tilde{H}^{\mu_1 \ldots \mu_{D-1}} + M \epsilon^{\mu_1 \mu_2 \ldots \mu_D} A_{\mu_1} \partial_{\mu_2} H_{\mu_3 \ldots \mu_D} \Big] \Big\},
\end{aligned}
\tag{2.7}
$$

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[1] Let ω be a p-form. Then there is a $(p+1)$-form α, a $(p-1)$-form β and a harmonic p-form h (i.e., obeying $\delta h = 0 = dh$) such that

$$
\omega = \delta\alpha + d\beta + h.
$$

We can restrict ourselves to the topologically trivial space Ω for which there are no harmonic forms. This is equivalent to saying that each p-form ω obeying $d\omega = 0$ is of the form $\omega = d\beta$ (Poincare's lemma) and we say Ω has trivial (co)homology. From now on we assume that the harmonic form is absent: $h = 0$.

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where we have defined

$$\tilde{H}^{\mu_1\cdots\mu_{D-1}} = \partial^{\mu_1}H^{\mu_2\cdots\mu_{D-1}} - \partial^{\mu_2}H^{\mu_1\mu_3\cdots\mu_{D-1}} + \ldots + (-1)^D\partial_{\mu_{D-1}}H^{\mu_1\cdots\mu_{D-2}}. \quad (2.8)$$

This result is a generalization of [20] for $D = 3$ and coincides with the result of [24, 2].

Then we obtain the equivalent Lagrangian with the mixed term between A_μ and $H_{\mu_1\ldots\mu_{D-2}}$:

$$
\begin{aligned}
\mathcal{L}_{Th'''} &= \bar{\psi}^j i\gamma^\mu D_\mu[A]\psi^j - m_j\bar{\psi}^j\psi^j \\
&\quad + \frac{(-1)^D}{2(D-1)}\tilde{H}_{\mu_1\ldots\mu_{D-1}}\tilde{H}^{\mu_1\cdots\mu_{D-1}} + M\epsilon^{\mu_1\cdots\mu_D}A_{\mu_1}\partial_{\mu_2}H_{\mu_3\ldots\mu_D} \\
&\quad - A_\mu\partial^\mu B + \frac{\xi}{2}B^2,
\end{aligned}
\quad (2.9)
$$

where $D_\mu[A]$ is the covariant derivative:

$$D_\mu[A] = \partial_\mu - i\frac{g}{\sqrt{N}}A_\mu. \quad (2.10)$$

Integrating out the fermion field $\bar{\psi}, \psi$, we thus obtain the bosonised action of the Thirring model:

$$
\begin{aligned}
S_B &= \sum_{j=1}^N \ln\det[i\gamma^\mu D_\mu[A] + m_j] \\
&\quad + \int d^D x\left[\frac{(-1)^D}{2(D-1)}\tilde{H}_{\mu_1\ldots\mu_{D-1}}\tilde{H}^{\mu_1\cdots\mu_{D-1}} + M\epsilon^{\mu_1\cdots\mu_D}A_{\mu_1}\partial_{\mu_2}H_{\mu_3\ldots\mu_D}\right. \\
&\quad \left. - A_\mu\partial^\mu B + \frac{\xi}{2}B^2\right].
\end{aligned}
\quad (2.11)
$$

To see the origin of the field $H_{\mu_1\ldots\mu_{D-2}}$, we integrate out the gauge field. Then, we obtain the partition function:

$$
\begin{aligned}
Z &= \int \mathcal{D}B\int \mathcal{D}\bar{\psi}\int \mathcal{D}\psi\int \mathcal{D}H_{\mu_1\ldots\mu_{D-2}} \\
&\quad \times\delta\left(\frac{g}{\sqrt{N}}\bar{\psi}^j\gamma^\mu\psi^j - M\epsilon^{\mu\mu_2\cdots\mu_D}\partial_{\mu_2}H_{\mu_3\ldots\mu_D} - \partial^\mu B\right) \\
&\quad \times\exp\left\{i\int d^D x\left[\bar{\psi}^j i\gamma^\mu(\partial_\mu)\psi^j - m_j\bar{\psi}^j\psi^j\right.\right. \\
&\quad \left.\left. + \frac{(-1)^D}{2(D-1)}\tilde{H}_{\mu_1\ldots\mu_{D-1}}\tilde{H}^{\mu_1\cdots\mu_{D-1}} + \frac{\xi}{2}B^2\right]\right\}.
\end{aligned}
\quad (2.12)
$$

This implies that the dual field $H_{\mu_1\ldots\mu_{D-2}}$ is a composite of the fermion and antifermion.

The correspondence between the original Thirring model and the bosonized theory is generalized to the correlation function. We introduce the source b_μ for the current

$$\mathcal{J}_\mu = \bar{\psi}^j\gamma_\mu\psi^j. \quad (2.13)$$

Adding the source term $\mathcal{J}_\mu b^\mu$ to the original Lagrangian eq. (2.1):

$$\mathcal{L}_{Th}[b_\mu] \equiv \mathcal{L}_{Th} + \mathcal{J}_\mu b^\mu, \tag{2.14}$$

we obtain

$$\mathcal{L}_{Th'}[b_\mu] = \bar{\psi}^j i\gamma^\mu (\partial_\mu - i\frac{g}{\sqrt{N}}A_\mu - ib_\mu)\psi^j - m_j\bar{\psi}^j\psi^j + \frac{M^2}{2}(A_\mu)^2. \tag{2.15}$$

After introducing the BF field θ and shifting $A_\mu \to A_\mu - \frac{\sqrt{N}}{g}b_\mu$, we obtain

$$\begin{aligned}
\mathcal{L}_{Th''}[b_\mu] &= \bar{\psi}^j i\gamma^\mu(\partial_\mu - i\frac{g}{\sqrt{N}}A_\mu)\psi^j - m_j\bar{\psi}^j\psi^j + \frac{M^2}{2}(A_\mu - \frac{\sqrt{N}}{g}b_\mu - K_\mu)^2 \\
&\quad -A_\mu\partial^\mu B + \frac{\xi}{2}B^2 + i\partial^\mu\bar{C}\partial_\mu C.
\end{aligned} \tag{2.16}$$

Repeating the same steps as before, we arrive at

$$\mathcal{L}_{Th'''}[b_\mu] = \mathcal{L}_{Th'''} + \frac{\sqrt{N}}{g}M\epsilon^{\mu_1...\mu_D}b_{\mu_1}\partial_{\mu_2}H_{\mu_3...\mu_D}. \tag{2.17}$$

This leads to the equivalence of the partition function in the presence of the source b_μ:

$$Z_{Th}[b_\mu] = Z_{Bosonised}[b_\mu]. \tag{2.18}$$

Therefore the connected correlation function has the following correspondence between the Thirring model and the bosonised theory with the action S_B:

$$\langle \mathcal{J}_{\mu_1};...;\mathcal{J}_{\mu_n}\rangle_{Th} = \langle \eta_{\mu_1};...;\eta_{\mu_n}\rangle_{Bosonised}, \tag{2.19}$$

where

$$\eta^{\mu_1} = \frac{1}{\sqrt{G/N}}\epsilon^{\mu_1...\mu_D}\partial_{\mu_2}H_{\mu_3...\mu_D}. \tag{2.20}$$

In the following we discuss how to integrate out the auxiliary field A_μ to obtain the bosonic theory which is written in terms of the field $H_{\mu_1...\mu_{D-2}}$ only.

3 (2+1)-dimensional case

In the three-dimensional case, $D = 3$,

$$\begin{aligned}
S_B &= \sum_{j=1}^{N}\ln\det[i\gamma^\mu D_\mu[A] + m_j] \\
&\quad + \int d^3x\left[-\frac{1}{4}\tilde{H}_{\mu\nu}\tilde{H}^{\mu\nu} + M\epsilon^{\mu\nu\rho}A_\mu\partial_\nu H_\rho - A_\mu\partial^\mu B + \frac{\xi}{2}B^2\right], \tag{3.1}
\end{aligned}$$

where

$$\tilde{H}_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu. \tag{3.2}$$

The Matthews-Salam determinant in eq. (3.1) is calculated with the aid of appropriate regularizations. There are various gauge-invariant regularization methods: 1) Pauli-Villars [25, 26, 27, 28], 2) lattice [29, 30], 3) analytic [31], 4) dimensional [32, 33], 5) Zavialov [34], 6) parity-invariant Pauli-Villars (variant of chiral gauge invariant Pauli-Villars by Frolov and Slavnov) [35], 7) high covariant derivative [36], 8) zeta-function [37]. However it should be remarked that the methods 1) and 2) give regulator dependent result for the Chern-Simons part. Appropriate choice of regularization leads to the regulator independent result, see for example [35]. In the massless limit, $m \to 0$, it is shown [26] up to one-loop that

$$\ln \det[i\gamma^\mu(\partial_\mu - i\frac{g}{\sqrt{N}}A_\mu) + m]$$

$$\to \quad \frac{sgn(m)}{N}\frac{i}{16\pi}g^2 \int d^3x \epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + I_{PC}[A_\mu], \tag{3.3}$$

where $sgn(m)$ denotes the signature of m and the parity-conserving term is given by

$$I_{PC}[A_\mu] = \frac{1}{\pi^2}\zeta(\frac{3}{2})\int d^3x \left(\frac{g}{2\sqrt{N}}F_{\mu\nu}^2\right)^{3/2}. \tag{3.4}$$

Therefore the bosonization of the massless Thirring model in (2+1) dimensions would lead to highly complicated bosonic theory.

In what follows we consider the large fermion mass limit, $m \to \infty$. For the 2×2 gamma matrices corresponding to the two-component fermion, $\bar{\psi}, \psi$,

$$N \ln \det[i\gamma^\mu D_\mu[A] + m_j] - N \ln \det[i\gamma^\mu \partial_\mu + m_j]$$

$$= \quad N \text{Tr} \ln \left[1 + (i\gamma^\mu\partial_\mu + m_j)^{-1}\frac{g}{\sqrt{N}}\gamma^\mu A_\mu\right]$$

$$= \quad sgn(m_j)\frac{ig^2}{16\pi}\int d^3x\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho - \frac{g^2}{24\pi|m_j|}\int d^3x F_{\mu\nu}F^{\mu\nu} + \mathcal{O}\left(\frac{\partial^2}{|m_j|^2}\right) \tag{3.5}$$

where $sgn(m)$ denotes signature of the fermion mass m, $sgn(m) = m/|m|$. This is understood as follows. Note that

$$N \text{Tr} \ln \left[1 + (i\gamma^\mu\partial_\mu + m)^{-1}\frac{g}{\sqrt{N}}\gamma^\mu A_\mu\right] = \frac{1}{2}\int d^D x A^\mu(x)\Pi_{\mu\nu}(\partial; m)A^\nu(x) + ... \tag{3.6}$$

For $D = 3$, the vacuum polarization tensor is given by

$$\Pi_{\mu\nu}(\partial; m) = \left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\Pi_T(-\partial^2, m) + i\epsilon_{\mu\nu\rho}\partial^\rho\Pi_O(-\partial^2; m), \tag{3.7}$$

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where

$$\Pi_T(k^2; m) = -\frac{g^2}{2\pi} k^2 \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{[m^2 - \alpha(1-\alpha)k^2]^{1/2}}, \tag{3.8}$$

and

$$\Pi_O(k^2; m) = -\frac{g^2 m}{4\pi} \int_0^1 d\alpha \frac{1}{[m^2 - \alpha(1-\alpha)k^2]^{1/2}}. \tag{3.9}$$

In the large fermion mass limit, we obtain eq. (3.5).

Thus the Thirring model in the large fermion mass limit is equivalent to the bosonized theory with the interpolating Lagrangian:

$$
\begin{aligned}
\mathcal{L}_I[A_\mu, H_\mu] = & -\frac{1}{4} \tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} + \frac{M}{2} \epsilon^{\mu\nu\rho} \tilde{H}_{\mu\nu} A_\rho + \frac{i\theta_{CS}}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho \\
& -\frac{1}{N} \sum_{j=1}^N \frac{GM^2}{24\pi |m_j|} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + \mathcal{O}\left(\frac{\partial^2}{m_j^2}\right), \quad (3.10)
\end{aligned}
$$

where

$$\theta_{CS} = \frac{1}{N} \sum_{j=1}^N sgn(m_j) \frac{GM^2}{4\pi}. \tag{3.11}$$

Integrating out the $\tilde{H}_{\mu\nu}$ field via $H_\rho^* := \frac{1}{2}\epsilon_{\mu\nu\rho}\tilde{H}^{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\mu H^\nu$ in the interpolating Lagrangian, we obtain the self-dual Lagrangian in the same sense as used in Deser and Jackiw [21]:

$$\mathcal{L}_{SD}[A_\mu] = \frac{M^2}{2} A^\mu A_\mu + \frac{i\theta_{CS}}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho, \tag{3.12}$$

up to the lowest order of $1/m$. This implies that the Thirring model is equivalent to the self-dual model with the Lagrangian $\mathcal{L}_{SD}$ in the lowest order of $1/m$.

We notice that the interpolating Lagrangian we have just obtained is essentially equivalent to the master Lagrangian of Deser and Jackiw [21]. The master Lagrangian for A_μ and H_μ^* is given by

$$\mathcal{L}_{DJ} = \frac{1}{2} A^\mu A_\mu - A_\mu H^{\mu*} + \frac{1}{2} \tilde{m} H^{\mu*} H_\mu. \tag{3.13}$$

Note that the role of the auxiliary field A_μ and the new field H_μ is interchanged in our interpolating Lagrangian compared with that in [18] based on the master Lagrangian of Deser and Jackiw. Hence the integration over the auxiliary field is non-trivial in our interpolating Lagrangian.

If $m_j = m$ for all $j = 1, ..., N$, then the original Lagrangian (2.1) has $O(N)$ symmetry. If we adopt the fermion mass term such that

$$m_j = \begin{cases} m & (j = 1, ..., N - k) \\ -m & (j = N - k + 1, ..., N) \end{cases}, \tag{3.14}$$

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the Lagrangian has $O(N-k) \times O(k)$ symmetry. For this mass term, we obtain

$$\theta_{CS} = \left(1 - 2\frac{k}{N}\right)\frac{GM^2}{4\pi}. \tag{3.15}$$

Hence, if we take $k = N/2$ in the fermion mass term, the theory has $O(N/2) \times O(N/2)$ symmetry and $\theta_{CS} = 0$ follows, although the kinetic term for the field A_μ in the next-to-leading order is unchanged. See the Vafa-Witten argument [38]. A similar situation occurs in the formulation which uses the 4-component fermion with 4×4 gamma matrices where the Chern-Simons term in the determinant disappear, since $\mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\rho) = 0$. In this case, the Thirring model is equivalent to the bosonised theory with the Lagrangian:

$$\begin{aligned}
\mathcal{L}_{I'} &= -\frac{1}{4}\tilde{H}_{\mu\nu}\tilde{H}^{\mu\nu} + M\epsilon^{\mu\nu\rho}A_\mu\partial_\nu H_\rho - \frac{g^2}{24\pi|m|}F_{\mu\nu}F^{\mu\nu} \\
&\quad - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \mathcal{O}\left(\frac{\partial^2}{|m|^2}\right).
\end{aligned} \tag{3.16}$$

Such a case is discussed in the final section.

Now we proceed to the step of integrating out the gauge field A_μ, assuming $\theta_{CS} \neq 0$ and the fermion mass pattern (3.14), i.e., $|m_j| = m$ for all j. By defining

$$J^\mu(x) = M\epsilon^{\mu\nu\rho}\partial_\nu H_\rho(x) = \frac{M}{2}\epsilon^{\mu\nu\rho}\tilde{H}_{\mu\nu}(x), \tag{3.17}$$

and

$$K^{\mu\nu}(x,y) = \left[i\theta_{CS}\epsilon^{\mu\nu\rho}\partial_\rho + \frac{1}{\xi}\partial^\mu\partial^\nu + \frac{g^2}{6\pi|m|}(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\right]\delta(x-y), \tag{3.18}$$

the interpolating action is written in the form:

$$\begin{aligned}
S_I &= \int d^3x\left[-\frac{1}{4}\tilde{H}_{\mu\nu}\tilde{H}^{\mu\nu} + A_\mu(x)J^\mu(x)\right] \\
&\quad + \int d^3x \int d^3y \frac{1}{2}A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y) + \mathcal{O}\left(\frac{\partial^2}{m^2}\right).
\end{aligned} \tag{3.19}$$

The A_μ integration in the interpolating action is performed by using

$$\begin{aligned}
&\int \mathcal{D}A_\mu \exp\left\{i\int d^3x \int d^3y \frac{1}{2}A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y) + i\int d^3x A_\mu(x)J^\mu(x)\right\} \\
&= \exp\left\{i\int d^3x \int d^3y \frac{1}{2}J^\mu(x)K^{-1}_{\mu\nu}(x,y)J^\nu(y)\right\},
\end{aligned} \tag{3.20}$$

up to a constant which is independent of the field variable. Here the inverse is obtained as

$$\begin{aligned}
K^{-1}_{\mu\nu}(x,y) &= \frac{i}{\theta_{CS}}\epsilon_{\mu\nu\rho}\partial^\rho\Delta^{(1)}(x,y) + \xi\partial_\mu\partial_\nu\Delta^{(2)}(x,y) \\
&\quad + \frac{g^2}{6\pi\theta_{CS}^2|m|}\left(g_{\mu\nu} - \partial_\mu\partial_\nu\Delta^{(1)}(x,y)\right) + \mathcal{O}\left(\frac{1}{|m|^2}\right),
\end{aligned} \tag{3.21}$$

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where $\Delta^{(1)} = 1/\partial^2$ and $\Delta^{(2)} = 1/\partial^4$ in the sense $\partial^2 \Delta^{(1)}(x,y) = \delta(x-y)$ and $(\partial^2)^2 \Delta^{(2)}(x,y) = \delta(x-y)$. By using this formula, A_μ integration is performed:

$$J^\mu K_{\mu\nu}^{-1} J^\nu = \frac{M^2}{i\theta_{CS}} \epsilon^{\mu\nu\rho} H_\mu \partial_\nu H_\rho + \frac{g^2 M^2}{12\pi\theta_{CS}^2 |m|} \tilde{H}_{\mu\nu}^2 + \mathcal{O}\left(\frac{1}{|m|^2}\right), \qquad (3.22)$$

which is independent of the gauge-fixing parameter ξ in the original theory.

Thus we arrive at an effective bosonised Lagrangian for the dual field H_μ alone:

$$\mathcal{L}_{MCS} = -\frac{1}{4}\left(1 + \frac{g^2 M^2}{6\pi\theta_{CS}^2 |m|}\right) \tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} + \frac{iM^2}{2\theta_{CS}} \epsilon^{\mu\nu\rho} H_\mu \partial_\nu H_\rho$$
$$+ \mathcal{O}\left(\frac{\partial^2}{|m|^2}\right). \qquad (3.23)$$

Note that the gauge-parameter dependence has dropped out in the bosonized theory. In the interpolating Lagrangian $\mathcal{L}_I$ the gauge degree of freedom for the A_μ field is fixed by the gauge-fixing term $\frac{1}{2\xi}(\partial^\mu A_\mu)^2$. However there is an additional gauge symmetry for the new field H_μ: the Lagrangian $\mathcal{L}_I$ is invariant under the gauge transformation $\delta H_\mu = \partial_\mu \omega$ independently of A_μ, which leads to $\delta f_\mu = 0$ in the master Lagrangian. Therefore we must add a gauge-fixing term for the H_μ field to the bosonized Lagrangian $\mathcal{L}_{MCS}$.

Thus, to the leading order of $1/m$ expansion, the Thirring model partition function coincides with that of the Maxwell-Chern-Simons theory. This result agrees with that in [18] obtained up to the leading of $1/m$ where the less direct procedure is adopted to show this equivalence using the self-dual action by way of the interpolating action. Up to the next-to-leading order of $1/m$, we have shown that the equivalence between the low energy sector of a theory of 3-dimensional fermions interacting via a current-current term and gauge bosons with Maxwell-Chern-Simons term is preserved. The Thirring spin-1/2 fermion with the Thirring coupling g^2/N is equal to a spin-1 massive excitation with mass

$$\frac{\pi}{g^2}\left(1 - \frac{g^2 M^2}{6\pi\theta_{CS}^2 |m|}\right) + \mathcal{O}(\frac{1}{m^2}), \qquad (3.24)$$

in 2+1 dimensions. In 2+1 dimensions there is the following correspondence between the original Thirring model and the bosonized theory:

$$\bar{\psi}^j \gamma^\mu \psi^j \leftrightarrow \frac{1}{\sqrt{G/N}} \epsilon^{\mu\nu\rho} \partial_\nu H_\rho. \qquad (3.25)$$

Especially, for $D = 3$, the relation Eq (2.7) shows that the London action (without a kinetic term $F_{\mu\nu}^2$) for superconductivity:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi - M A_\mu)^2 = \frac{M^2}{2}(A_\mu - M^{-1}\partial_\mu \phi)^2, \qquad (3.26)$$

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is equivalent to the Chern-Simons term:

$$\mathcal{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} - M\epsilon^{\mu\nu\rho}A_\mu\partial_\nu H_\rho. \tag{3.27}$$

This fact was already pointed out in [20]. The missing kinetic term for A_μ is generated through the radiative correction as shown above. The Meissner effect in superconductivity is nothing but the Higgs phenomenon: the photon (massless gauge field) becomes massive gauge boson by absorbing the massless Nambu-Goldstone boson (scalar mode). The mixed Chern-Simons action does not break the parity in sharp contrast to the ordinary Chern-Simons term. Therefore this model may be a candidate for the high-T_c superconductivity without parity violation, as suggested in [20].

Finally we wish to point out that, in the large m limit, the Thirring model is also equivalent to the Chern-Simons-Higgs model up to the leading order of $1/m$ and to the Maxwell-Chern-Simons-Higgs model up to the next-to-leading order of $1/m$, since

$$\begin{aligned}
\mathcal{L}_{CSH} &= (D_\mu\varphi)^\dagger(D^\mu\varphi) + \frac{i\theta_{CS}}{4}\int d^3x\,\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho \\
&\quad -\frac{g^2}{24\pi|m|}\int d^3x\,F_{\mu\nu}F^{\mu\nu} + \mathcal{O}\left(\frac{\partial^2}{|m|^2}\right),
\end{aligned} \tag{3.28}$$

apart from the gauge-fixing term. This result is consistent with the assertion of Deser and Yang [39].

4   (1+1)-dimensional case

The bosonization of the (1+1)-dimensional Thirring model has a long history and is well known. Therefore there are a lot of references on bosonization which can not be exhausted, see for example [7, 8, 9, 10] based on the canonical formalism and [11, 12, 13, 14, 15, 16] based on the path integral formalism. In this section we reproduce the well-known result based on our method.

For $D = 2$, it is easy to show that the bosonized action is given by

$$\begin{aligned}
S_B &= N\ln\det[i\gamma^\mu D_\mu[A] + m] \\
&\quad + \int d^2x\left[\frac{1}{2}(\partial_\mu H)^2 + M\epsilon^{\mu\nu}A_\mu\partial_\nu H - A_\mu\partial^\mu B + \frac{\xi}{2}B^2\right].
\end{aligned} \tag{4.1}$$

The determinant is calculated as

$$\begin{aligned}
&N\ln\frac{\det[i\gamma^\mu D_\mu[A] + m]}{\det[i\gamma^\mu\partial_\mu + m]} \\
&= \frac{1}{2}\int d^2x\,A^\mu(x)\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\Pi_T(-\partial^2; m)A^\nu(x),
\end{aligned} \tag{4.2}$$

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where

$$\Pi_T(k^2; m) = -\frac{g^2}{\pi} k^2 \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{m^2 - \alpha(1-\alpha)k^2}. \tag{4.3}$$

In the massless case, $m = 0$,

$$\Pi_T(k^2; m = 0) = \frac{g^2}{\pi}, \tag{4.4}$$

and hence

$$K_{\mu\nu} = \frac{g^2}{\pi} \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \frac{1}{\xi} \partial_\mu \partial_\nu, \tag{4.5}$$

whose inverse is given by

$$K_{\mu\nu}^{-1} = \frac{\pi}{g^2} \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \xi \frac{\partial_\mu \partial_\nu}{\partial^4}. \tag{4.6}$$

Putting

$$J^\mu = M \epsilon^{\mu\nu} \partial_\nu H, \tag{4.7}$$

and integrating out the A_μ field, we obtain the bosonised theory:

$$S_B = \int d^2x \frac{1}{2} \left(1 + \frac{\pi}{G} \right) (\partial_\mu H)^2. \tag{4.8}$$

The massless Thirring model in two dimensions is equivalent to the massless scalar field theory, as long as the action is bounded from below, i.e., $G := g^2/M^2 > 0$ or $G < -\pi$. For more details on the massless case, see reference [2].

On the other hand, in the large m limit, we find

$$\Pi_T(k^2; m) = -\frac{g^2}{\pi} \left[\frac{1}{6} \frac{k^2}{m^2} + \frac{1}{30} \frac{k^4}{m^4} \right] + \mathcal{O}(\frac{k^4}{m^4}), \tag{4.9}$$

and hence

$$K_{\mu\nu} = \frac{g^2}{6\pi} \frac{\partial^2}{m^2} \left(1 - \frac{1}{5} \frac{\partial^2}{m^2} \right) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \frac{1}{\xi} \partial_\mu \partial_\nu + \mathcal{O}(\frac{\partial^2}{m^2}). \tag{4.10}$$

The inverse is obtained as

$$K_{\mu\nu}^{-1} = \frac{6\pi}{g^2} \frac{m^2}{\partial^2} \left(1 + \frac{1}{5} \frac{\partial^2}{m^2} \right) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) + \xi \frac{\partial_\mu \partial_\nu}{\partial^4} + \mathcal{O}(\frac{\partial^2}{m^2}). \tag{4.11}$$

Hence we obtain the bosonized theory after integrating out the A_μ field:

$$S_B = \int d^2x \left[\frac{1}{2} \left(1 + \frac{6}{5} \frac{\pi}{G} \right) (\partial_\mu H)^2 - \frac{3\pi m^2}{G} H^2 \right] + \mathcal{O}(\frac{\partial^4}{m^2}). \tag{4.12}$$

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Thus in two dimensions the massive Thirring model with large mass $m \gg 1$ is equivalent to the scalar field theory with large mass:

$$m_H = \sqrt{\frac{6\pi m^2}{G}\left(1 + \frac{6}{5}\frac{\pi}{G}\right)}. \tag{4.13}$$

In the massive limit, the Thirring model is physically sensible for $G > 0$.

In two dimensions there is a correspondence between the Thirring model and the bosonised theory:

$$\bar{\psi}^j \gamma^\mu \psi^j \leftrightarrow \frac{1}{\sqrt{G/N}} \epsilon^{\mu\nu} \partial_\nu H. \tag{4.14}$$

The above results are reasonable as shown in the following. The massive Thirring model is not exactly soluble even in (1+1)-dimensions. However the (1+1)-dimensional massive Thirring model is equivalent to the sine-Gordon model [8]:

$$\mathcal{L}_{sG} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}[\cos(\beta\varphi) - 1], \tag{4.15}$$

if the following identifications are made between the two theories:

$$1 + \frac{G}{\pi} = \frac{4\pi}{\beta^2}, \tag{4.16}$$

$$\bar{\psi}\gamma^\mu\psi = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi, \tag{4.17}$$

$$m\bar{\psi}\psi = -\frac{\alpha}{\beta^2}\cos(\beta\varphi), \tag{4.18}$$

where a constant in the Lagrangian $\mathcal{L}_{sG}$ is adjusted so that the minimum of the energy density is zero.

The massless limit $m \to 0$ of the massive Thirring model corresponds to the limit $\alpha \to 0$ in the sine-Gordon model, i.e., a massless scalar field theory in agreement with the above result. On the other hand, the massive limit $m \to \infty$ corresponds to the limit $\beta \to 0$ (or $\alpha \to \infty$) which inevitably leads to the limit $G \to \infty$ as $\frac{\beta}{2\pi} \sim \frac{1}{\sqrt{G}}$. Hence in this limit the Lagrangian reduces to

$$\mathcal{L}_{sG} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\alpha}{2}\varphi^2 + \mathcal{O}(\alpha\beta^2\varphi^4). \tag{4.19}$$

Moreover the eq. (4.17) recovers the above correspondence relation eq. (4.14) for $N = 1$. Therefore the very massive limit $m \gg 1$ of the 2-dimensional Thirring model is equivalent to the massive free scalar field theory with mass $\sqrt{\alpha}$ which is identified with m_H given above.

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5 Conclusion and Discussion

In this paper we have investigated the bosonization of the multiflavor massive Thirring model in $D = d + 1 \geq 2$ dimensions, starting from a reformulation of the Thirring model as a gauge theory.

In (1+1) dimensions, we have reproduced the well known result [7, 8, 9]. In (2+1) dimensions we have found a novel interpolating Lagrangian:

$$\mathcal{L}_I[A_\mu, H_\mu] = -\frac{1}{4}\tilde{H}_{\mu\nu}\tilde{H}^{\mu\nu} + \frac{M}{2}\epsilon^{\mu\nu\rho}\tilde{H}_{\mu\nu}A_\rho + \mathcal{L}_G[A_\mu], \tag{5.1}$$

where $\mathcal{L}_G[A_\mu]$ is the Lagrangian for the gauge field A_μ generated from the fermion determinant up to the next-to-leading order in $1/|m_j|$:

$$\mathcal{L}_G[A_\mu] = \frac{i\theta_{CS}}{4}\epsilon^{\mu\nu\rho}A_\rho F_{\mu\nu} - \frac{g^2}{24\pi|m|}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}\left(\frac{\partial^2}{|m|^2}\right). \tag{5.2}$$

This interpolating Lagrangian is shown to interpolate between the massive Thirring model with the Lagrangian $\mathcal{L}_{Th}$ and the Maxwell-Chern-Simons theory with the Lagrangian:

$$\mathcal{L}_{MCS}[H_\mu] = -\frac{1}{4}\left(1 + \frac{G}{6\pi\theta_{CS}^2|m|}\right)\tilde{H}_{\mu\nu}\tilde{H}^{\mu\nu} + \frac{iM^2}{2\theta_{CS}}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu H_\rho. \tag{5.3}$$

In contrast to the previous interpolating Lagrangian [21, 18], our interpolating Lagrangian is invariant under the independent gauge transformations for two gauge fields, A_μ, H_μ:

$$\delta A_\mu = \partial_\mu\lambda, \quad \delta H_\mu = \partial_\mu\omega. \tag{5.4}$$

It is interesting to see that the weak coupling limit $G \downarrow 0$ (i.e., $\theta_{CS} \downarrow 0$) of the massive Thirring model is nothing but the topological theory, the Chern-Simons theory, in the leading order of $1/|m|$.

In the multiflavor Thirring case we have treated in this paper, a specific situation such that $\theta_{CS} = 0$ may occur for even number of 2-component fermions or 4-component fermions which is important in discussing the chiral symmetry breaking [3] (see eq. (3.15)), in contrast to the case considered in [18]. In such a case, the inverse of $K_{\mu\nu}$ is given by

$$K_{\mu\nu}^{-1}(x, y) = \xi\partial_\mu\partial_\nu\Delta^{(2)}(x, y) + \frac{3\pi|m|}{G}\Delta^{(1)}\left(g_{\mu\nu} - \partial_\mu\partial_\nu\Delta^{(1)}\right)(x, y). \tag{5.5}$$

This leads to the non-local Maxwell-like Lagrangian. In the multiflavor case, therefore, the exact bosonization of the massive Thirring model up to the next-to-leading order in $1/m$ is possible only when fermion masses are configured such that the coefficient θ_{CS} is nonzero.

In order to formulate the Thirring model as a gauge theory based on the BFV formalism, we need a kinetic term for the gauge field A_μ. Therefore the existence

of the next-to-leading order term in $\mathcal{L}_G[A]$ is essential in our formulation. The Chern-Simons theory and the Maxwell theory has completely different structure as a constraint system in the bosonization scenario. From this point of view, we consider the bosonization in a subsequent paper [40].

In dimensions $D \geq 4$, the Lagrangian $\mathcal{L}_G[A]$ for the gauge field A_μ coming from integrating out the fermion field becomes non-local, since

$$
\begin{aligned}
\Pi_T(k^2; m) &= -k^2 \frac{2\mathrm{tr}(1)\Gamma(2 - D/2)}{(4\pi)^{D/2}} \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{[\alpha(1-\alpha)k^2 + m^2]^{2-D/2}} \\
&= -\frac{\mathrm{tr}(1)\Gamma(2 - D/2)}{3(4\pi)^{D/2}} \frac{k^2}{m^{4-D}} {}_2F_1(2, 2 - \frac{D}{2}, \frac{5}{2}; -\frac{k^2}{4m^2}),
\end{aligned} \quad (5.6)
$$

where ${}_2F_1(a, b, c; z)$ is a hypergeometric function. This makes the bosonization in the exact sense rather difficult together with the reducibleness of the antisymmetric tensor gauge theory written in terms of $H_{\mu_3 \ldots \mu_D}$. This case will be discussed elsewhere.

Finally we discuss how to extend our method into the non-Abelian case. First of all, we observe that the following master Lagrangian is equivalent to the interpolating Lagrangian (2.3) apart from the gauge-fixing and the ghost terms:

$$
\mathcal{L}_M[A_\mu, H_\mu, K_\mu] = \frac{M^2}{2}(A_\mu - K_\mu)^2 + \frac{1}{2}\epsilon^{\mu_1 \ldots \mu_D} H_{\mu_3 \ldots \mu_D} F_{\mu_1 \mu_2}[K] + \mathcal{L}_G[A_\mu], \quad (5.7)
$$

where $F_{\mu\nu}[K]$ is the field strength for K_μ defined by $F_{\mu\nu}[K] := \partial_\mu K_\nu - \partial_\nu K_\mu$. By introducing the vector field f_μ as in section 2, this Lagrangian is cast into

$$
\mathcal{L}_M = -\frac{1}{2}(f_\mu)^2 + M f^\mu (A_\mu - K_\mu) + \frac{1}{2}\epsilon^{\mu_1 \ldots \mu_D} H_{\mu_3 \ldots \mu_D} F_{\mu_1 \mu_2}[K] + \mathcal{L}_G[A_\mu]. \quad (5.8)
$$

Integrating out the field K_μ, we obtain the constraint $f^{\mu_1} = \epsilon^{\mu_1 \ldots \mu_D} \partial_{\mu_2} H_{\mu_3 \ldots \mu_D}$. Therefore the Lagrangian (5.7) reproduces the interpolating Lagrangian (2.9) for arbitrary dimension and especially (5.1) for $D = 3$.

Next, we point out that, by introducing the scalar field

$$
\varphi = \sqrt{N} \frac{V}{\sqrt{2}} e^{i\theta/V}, \quad (V = M/g = 1/\sqrt{G}), \quad (5.9)
$$

and integrating out the fermion field, the Thirring model as a gauge theory can be regarded with the gauged non-linear sigma model:

$$
\mathcal{L}_H = (D_\mu[A]\varphi)^\dagger (D^\mu[A]\varphi) + \mathcal{L}_G[A], \quad (5.10)
$$

with a local constraint: $\varphi(x)\varphi^*(x) = \frac{N}{2G}$. Actually, after redefinition of the field variable, another form of the master Lagrangian (5.7) is obtained:

$$
\mathcal{L}_M[A_\mu, H_\mu, V_\mu] = \frac{M^2}{2}(V_\mu)^2 + \frac{1}{2}\epsilon^{\mu_1 \ldots \mu_D} H_{\mu_3 \ldots \mu_D} F_{\mu_1 \mu_2}[V + A] + \mathcal{L}_G[A_\mu], \quad (5.11)
$$

which is shown to be at least classically equivalent to the non-linear sigma model with the Lagrangian $\mathcal{L}_H$. Indeed the master Lagrangian (5.11) has the local

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gauge invariance (5.4). It is easy to extend the master Lagrangian (5.7) or (5.11) into the non-Abelian case. However the quantum nature of the theory produces subtle problems in bosonization. The bosonization of a non-Abelian version of the massive Thirring model will be given in a subsequent paper [40].

Acknowledgments

Note added

Quite recently, a non-Abelian version of the massive Thirring model in (2+1)-dimensions was bosonized in [41] by following the same strategy as the Abelian case [18]. After having submitted this paper for publication, the author was informed that the basic idea of introducing the gauge degrees of freedom was already introduced by Burgess et al [42], although he was not aware of existence of such works. After our manuscript has been submitted to the preprint database (hep-th/9502100), a few interesting papers [43] appeared where the equivalence is shown in the different method by using the same interpolating Lagrangian as derived in this paper. However they do not derive the interpolating Lagrangian and consider it as a starting point without discussing its origin.

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